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LETTER TO THE EDITOR

Anisotropic Lifshitz point at $O(\epsilon_L^2)$

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Abstract
We present the critical exponents ν_{L2}, η_{L2} and γ_L for an m-axial Lifshitz point at second order in an ϵ_L expansion. We introduce a constraint involving the loop momenta along the m-dimensional subspace in order to perform two- and three-loop integrals. The results are valid in the range $0 \leqslant m < d$. The case $m = 0$ corresponds to the usual Ising-like critical behaviour.

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Lifshitz multicritical points appear at the confluence of a disordered phase, a uniformly ordered phase and a modulated ordered phase [1, 2]. The spatially modulated phase is characterized by a fixed equilibrium wavevector $\vec{k}_0$. In this phase, $\vec{k}_0$ goes continuously to zero as the system approaches the Lifshitz point. If this wavevector has m components, the critical system under consideration presents an m-fold Lifshitz critical behaviour. This sort of critical behaviour is present in a variety of real physical systems including high-T_c superconductors [3–5], ferroelectric liquid crystals [6, 7], magnetic compounds and alloys [8–10], among others.

In magnetic systems [11], the m-fold Lifshitz point can be described by a spin-$\frac{1}{2}$ Ising model on a d-dimensional lattice with nearest-neighbour ferromagnetic interactions as well as next-nearest-neighbour competing antiferromagnetic couplings along m directions. This system can be described in a field-theoretic setting using a modified ϕ^4 theory with higher-order derivative terms, which arises as an effect of the competition along the m directions. The Lifshitz universality class is defined by the parameters (N, d, m), where N is the number of components of the order parameter, d is the space dimension of the system and m is the number of competing directions.

Other examples of field theories containing higher derivative terms have been investigated in different physical scenarios. In cosmology, the recently proposed model known as 'k-inflation' describes inflation driven by higher-order kinetic terms for the inflaton scalar field [12]. Another instance which arises in quantum field theory in curved spacetime is the quantization of scalar fields with a high-frequency dispersion relation around a classical gravitational background [13, 14]. In this case, the higher-order term accounts for deviations from Lorentz invariance. The modified dispersion relation might arise from an unspecified modification of the short-distance structure of spacetime. A further generalization of this idea is to modify the large-distance structure of spacetime, allowing higher derivative terms, breaking Lorentz invariance in the infrared regime as well [15]. Thus a better comprehension of how to calculate arbitrary loop corrections for the Lifshitz critical behaviour should give a clue about the proper perturbative treatment needed for a general higher-order field theory.

In this paper we generalize the method recently developed for the $m = 1$ case [16, 17] to calculate the critical exponents η_{L2}, ν_{L2} and γ_L using renormalization group techniques and the ϵ_L-expansion up to $O(\epsilon_L^2)$, where $\epsilon_L = 4 + \frac{m}{2} - d$ is the expansion parameter in the perturbative analysis. We recover the results for the $m = 1$ case obtained in [17] and show for the first time that the Lifshitz critical behaviour reduces to the Ising-like one for $m = 0$. Thus, the Ising-like universality class (N, d) is contained in a nontrivial way in the Lifshitz (N, d, m).

We start with the bare Lagrangian associated with the Lifshitz critical behaviour. It can be written as a modified ϕ^4 field theory expressed in the following form:

$$L = \frac{1}{2}|\nabla_m^2 \phi_0|^2 + \frac{1}{2}|\nabla_{(d-m)} \phi_0|^2 + \delta_0 \frac{1}{2}|\nabla_m \phi_0|^2 + \frac{1}{2} t_0 \phi_0^2 + \frac{1}{4!}\lambda_0 \phi_0^4. \qquad (1)$$

The quartic dependence on the momenta along the m directions will be manifest in the free propagator. Here we will consider the system at the Lifshitz critical point, defined by the values $\delta_0 = t_0 = 0$. In order to compute the critical exponents, we need to calculate some Feynman diagrams, namely I_2, I_3, I_4 and I_5 [17, 18]. Setting $t_0 = \delta_0 = 0$,

$$I_2 = \int \frac{d^{d-m}q\, d^m k}{\left[\left((k + K')^2\right)^2 + (q + P)^2\right]\left((k^2)^2 + q^2\right)} \qquad (2)$$

is the one-loop integral contributing to the four-point function,

$$I_3 = \int \frac{d^{d-m}q_1\, d^{d-m}q_2\, d^m k_1\, d^m k_2}{\left(q_1^2 + (k_1^2)^2\right)\left(q_2^2 + (k_2^2)^2\right)\left[(q_1 + q_2 + p)^2 + \left((k_1 + k_2 + k')^2\right)^2\right]} \qquad (3)$$

is the two-loop 'sunset' Feynman diagram of the two-point function,

$$I_4 = \int \frac{d^{d-m}q_1\, d^{d-m}q_2\, d^m k_1\, d^m k_2}{\left(q_1^2 + (k_1^2)^2\right)\left((P - q_1)^2 + \left((K' - k_1)^2\right)^2\right)\left(q_2^2 + (k_2^2)^2\right)} \times \frac{1}{(q_1 - q_2 + p_3)^2 + \left((k_1 - k_2 + k_3')^2\right)^2} \qquad (4)$$

is one of the two-loop graphs which will contribute to the fixed point and

$$I_5 = \int \frac{d^{d-m}q_1\, d^{d-m}q_2\, d^{d-m}q_3\, d^m k_1\, d^m k_2\, d^m k_3}{\left(q_1^2 + (k_1^2)^2\right)\left(q_2^2 + (k_2^2)^2\right)\left(q_3^2 + (k_3^2)^2\right)\left[(q_1 + q_2 - p)^2 + \left((k_1 + k_2 - k')^2\right)^2\right]} \times \frac{1}{(q_1 + q_3 - p)^2 + \left((k_1 + k_3 - k')^2\right)^2} \qquad (5)$$

is the three-loop diagram contributing to the two-point vertex function. We then choose a special symmetry point in order to simplify the integrals. We set the external momenta at the quartic directions equal to zero, i.e. $k' = k_1' = k_2' = k_3' = 0$, and $K' = k_1' + k_2'$. In addition, for the four-point vertex, the external momenta along the quadratic directions are chosen as $p_i \cdot p_j = \frac{\kappa^2}{4}(4\delta_{ij} - 1)$, where p_1, p_2, p_3 are the independent external momenta, and $P = p_1 + p_2$. We fix the momentum scale of the two-point function through $p^2 = \kappa^2 = 1$. We shall use normalization conditions for the vertex functions along with dimensional regularization for the calculation of the Feynman diagrams.

Let us find the one-loop integral I_2. With our choice of the symmetry point, and introducing two Schwinger parameters, we obtain for I_2

$$\int \frac{\mathrm{d}^{d-m}q\,\mathrm{d}^m k}{\left((k^2)^2+(q+P)^2\right)\left((k^2)^2+q^2\right)} = \int_0^\infty \int_0^\infty \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \left(\int \mathrm{d}^m k\ \exp(-(\alpha_1+\alpha_2)(k^2)^2)\right) \times \int \mathrm{d}^{d-m}q\ \exp(-(\alpha_1+\alpha_2)q^2 - 2\alpha_2 q\cdot P - \alpha_2 P^2). \tag{6}$$

The $\vec{q}$ integral can be performed to give

$$\int \mathrm{d}^{d-m}q\ \exp(-(\alpha_1+\alpha_2)q^2 - 2\alpha_2 q\cdot P - \alpha_2 P^2) = \frac{1}{2} S_{d-m}\Gamma\left(\frac{d-m}{2}\right)(\alpha_1+\alpha_2)^{-\frac{d-m}{2}}\ \exp\left(-\frac{\alpha_1\alpha_2 P^2}{\alpha_1+\alpha_2}\right). \tag{7}$$

For the $\vec{k}$ integral we perform the change of variables $r^2 = k_1^2 + \cdots + k_m^2$. Now take $z = r^4$. The integral turns out to be

$$\int \mathrm{d}^m k\ \exp(-(\alpha_1+\alpha_2)(k^2)^2) = \left(\frac{1}{4}S_m\right)\Gamma\left(\frac{m}{4}\right)(\alpha_1+\alpha_2)^{-\frac{m}{4}}. \tag{8}$$

Using equations (7) and (8), I_2 reads

$$I_2 = \frac{1}{2}S_{d-m}\left(\frac{1}{4}S_m\right)\Gamma\left(\frac{d-m}{2}\right)\Gamma\left(\frac{m}{4}\right) \times \int_0^\infty\int_0^\infty \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\ \exp\left(-\frac{\alpha_1\alpha_2 P^2}{\alpha_1+\alpha_2}\right)(\alpha_1+\alpha_2)^{-(\frac{d}{2}-\frac{m}{4})}. \tag{9}$$

The remaining parametric integrals can be solved by a change of variables followed by a rescaling [19]. The integral is proportional to $(P^2)^{-\frac{\epsilon_L}{2}}$. Now we can set $P^2 = \kappa^2 = 1$. Using the identity

$$\Gamma(a+bx) = \Gamma(a)\left[1 + b\,x\,\psi(a) + \mathrm{O}(x^2)\right] \tag{10}$$

where $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z)$, one is able to perform the ϵ_L-expansion when the gamma functions have non-integer arguments. Altogether, the final result for I_2 is

$$I_2 = \left[\frac{1}{4}S_m S_{d-m}\Gamma\left(2-\frac{m}{4}\right)\Gamma\left(\frac{m}{4}\right)\right]\frac{1}{\epsilon_L}\left(1 + [i_2]_m\,\epsilon_L\right) \tag{11}$$

where $[i_2]_m = 1 + \frac{1}{2}(\psi(1) - \psi(2-\frac{m}{4}))$. From now on, we shall absorb the factor inside the brackets in equation (11) in the definition of the coupling constant [18]. Then the redefined integral is

$$I_2 = \frac{1}{\epsilon_L}\left(1 + [i_2]_m\,\epsilon_L\right). \tag{12}$$

Now we shall discuss the two- and three-loop integrals. We introduce a constraint among the loop momenta in different subdiagrams, along the quartic directions only [17]. We wish to highlight this approximation here by calculating the integral I_4 for $m \neq 8$.

After our choice for the external momenta along the quartic directions, we can write I_4 in the following way:

$$I_4 = \int \frac{\mathrm{d}^{d-m}q_1\,\mathrm{d}^m k_1}{\left(q_1^2+(k_1^2)^2\right)\left((P-q_1)^2+(k_1^2)^2\right)} \int \frac{\mathrm{d}^{d-m}q_2\,\mathrm{d}^m k_2}{\left(q_2^2+(k_2^2)^2\right)\left[(q_1-q_2+p_3)^2+((k_1+k_2)^2)^2\right]} \tag{13}$$

where we have changed variables from $k_2 \to -k_2$. We integrate first over the subdiagram q_2, k_2. In order to integrate over $\vec{k}_2$ we introduce a constraint relating $\vec{k}_1$ to $\vec{k}_2$ inside this subdiagram; i.e., $\vec{k}_1$ is fixed into the second integral in equation (13). If the relation between the two loop momenta is of the form $\vec{k}_1 = -\alpha\,\vec{k}_2$ we can solve the integral in terms of a product of gamma functions and a hypergeometric function. The value $\alpha = 2$ is singled out when we demand that the integral is given in terms of gamma functions only. This is a natural generalization of the $m = 1$ case [17]. Using Schwinger's parametrization and setting $\vec{k}_1 = -2\vec{k}_2$ in the second integral in equation (13) we find

$$I_4 = I_2 \int \frac{\mathrm{d}^{d-m}q_1\,\mathrm{d}^m k_1}{\left(q_1^2 + (k_1^2)^2\right)\left((P - q_1)^2 + (k_1^2)^2\right)} \frac{1}{\left[(q_1 + p_3)^2\right]^{\frac{\epsilon_L}{2}}}. \tag{14}$$

Performing the integral over k_1 we obtain

$$I_4 = I_2 \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^{d-m}q_1}{\left(q_1^2 - 2z\,P\cdot q_1 + zP^2\right)^{2-\frac{m}{4}} \left[(q_1 + p_3)^2\right]^{\frac{\epsilon_L}{2}}}. \tag{15}$$

Using a Feynman parameter the integral turns out to be

$$I_4 = \frac{1}{2} I_2 \left(1 - \frac{\epsilon_L}{2}\,\psi\left(2 - \frac{m}{4}\right)\right) \frac{\Gamma(\epsilon_L)}{\Gamma\left(\frac{\epsilon_L}{2}\right)} \int_0^1 \mathrm{d}y\, y^{1-\frac{m}{4}}\,(1-y)^{\frac{1}{2}\epsilon_L - 1}$$
$$\times \int_0^1 \mathrm{d}z \left[yz(1 - yz)P^2 + y(1-y)p_3^2 - 2yz(1-y)p_3\cdot P\right]^{-\epsilon_L}. \tag{16}$$

The integral over y is singular at $y = 1$ when $\epsilon_L = 0$. We only need to replace the value $y = 1$ inside the integral over z [17, 18], and integrate over y afterwards, obtaining

$$I_4 = \frac{1}{2\epsilon_L^2}\left(1 + 3\,[i_2]_m \epsilon_L\right). \tag{17}$$

The integrals I_3' and I_5' can be solved using a similar reasoning. They are given by

$$I_3' = -\frac{1}{8-m}\,\frac{1}{\epsilon_L}\left[1 + \left([i_2]_m + \frac{3}{4 - \frac{m}{2}}\right)\epsilon_L\right] \tag{18}$$

$$I_5' = -\frac{1}{3\left(2 - \frac{m}{4}\right)}\,\frac{1}{\epsilon_L^2}\left[1 + 2\left([i_2]_m + \frac{1}{2 - \frac{m}{4}}\right)\epsilon_L\right]. \tag{19}$$

Note that the leading singularities for I_2 and I_4 are the same as their analogous integrals in the pure ϕ^4 theory. However, I_3' and I_5' do not have the same leading singularities for they include a factor of $\frac{1}{(2-\frac{m}{4})}$. We then introduce a weight factor for I_3' and I_5', namely $(1 - \frac{m}{8})$, so that they have the same leading singularities as in the pure ϕ^4 theory. This has the advantage of allowing a smooth transition to the Ising-like case ($m = 0$) from the general Lifshitz anisotropic critical behaviour ($m \neq 8$) as we shall see next.

The fixed point at two-loop level is given by

$$u^* = \frac{6}{8+N}\,\epsilon_L\left\{1 + \epsilon_L\left[\left(\frac{4(5N+22)}{(8+N)^2} - 1\right)[i_2]_m - \frac{(2+N)}{(8+N)^2}\right]\right\}. \tag{20}$$

With this fixed point one readily obtains the critical exponents η_{L2} and ν_{L2}:

$$\eta_{L2} = \frac{1}{2}\epsilon_L^2\,\frac{2+N}{(8+N)^2} + \epsilon_L^3\,\frac{(2+N)}{(8+N)^2}\left[\left(\frac{4(5N+22)}{(8+N)^2} - \frac{1}{2}\right)[i_2]_m + \frac{1}{8-m} - \frac{2+N}{(8+N)^2}\right] \tag{21}$$

$$\nu_{L2} = \frac{1}{2} + \frac{1}{4}\epsilon_L\,\frac{2+N}{8+N} + \frac{1}{8}\,\frac{(2+N)}{(8+N)^3}\left[2(14N+40)\,[i_2]_m - 2(2+N) + (8+N)(3+N)\right]\epsilon_L^2. \tag{22}$$

Using the scaling law $\gamma_L = \nu_{L2}(2 - \eta_{L2})$, the exponent γ_L is

$$\gamma_L = 1 + \frac{1}{2}\epsilon_L \frac{2+N}{8+N} + \frac{1}{4}\frac{(2+N)}{(8+N)^3}\left[12 + 8N + N^2 + 4\,[i_2]_m\,(20+7N)\right]\epsilon_L^2. \tag{23}$$

It should be emphasized that $[i_2]_m$ is a universal amount, for the dependence on m is encoded in such a quantity. The parameter m only appears in a explicit way in the $O(\epsilon_L^3)$ contribution to the index η_{L2}. To our knowledge, the explicit dependence on m is obtained for the first time at $O(\epsilon_L^3)$ for η_{L2}. When setting $(m = 1)$ in the formulae above, we recover the exponents previously reported in [17]. As discussed there, the two-loop calculation $(N = 1)$ in three dimensions yields $\gamma_L = 1.45$, in a nice agreement with the numerical Monte Carlo simulation $\gamma_L = 1.4 \pm 0.06$.

The amazing fact obtained using the method outlined here is that the critical exponents reduce to the Ising-like ones when $m = 0$, for $\epsilon_L \to \epsilon = 4 - d$. This means that the universality class for the m-fold Lifshitz point includes the Ising-like one for this particular value of m in a nontrivial way. This provides a unified description of the anisotropic Lifshitz critical behaviour ($m \neq 8$, $d \neq m$). This is the first time that an isotropic behaviour $(m = 0)$ has been recoverable from the most general anisotropic Lifshitz criticality.

Note that our result for the exponent η_{L2} is in agreement with Mukamel's [20] at $O(\epsilon_L^2)$ and is independent of m at this order. It should not be surprising that the approach fails to describe the $d = m = 8$ case, for the exponent η_{L2} is divergent as can be seen from equation (21). The approximation made is not suitable for general isotropic cases $d = m \neq 8$ as well, since there is no longer any preferred direction. Another treatment should be employed to obtain information along the m-dimensional competition axes, since the symmetry point used here is not suitable to find quantities along the competing directions.

All the results in this paper follow from expanding the theory around its upper critical dimension. The constraint introduced along the m-dimensional subspace is equivalent to expanding around the theory without competition, by varying the space dimension d, with m kept fixed. This is the main difference between the approach described here and other proposals [21–23]. The first calculation for the critical index η_{L2} for general $m(\neq d)$ was performed [20] using the method of momentum shell integration based on the Hamiltonian formalism. Later [21], the computation of this critical index for the $m = 2, 6$ cases was done using a cutoff in coordinate space. It turned out that the two results do not agree. Recently, a different field-theoretic method has been proposed, based on the perturbative expansion around the number of the m competing directions [22] using dimensional regularization. Another technique using test functions on coordinate space together with dimensional regularization was developed in [23]. The three latter methods rely, in one way or another, on coordinate space computations. We believe that this is why they all agree among each other for the $m = 2, 6$ cases. On the other hand, the two different evaluations performed entirely in momentum space, namely Mukamel's and ours, yielded the same results. This suggests that calculations performed in momentum space and coordinate space are inequivalent, as far as the Lifshitz critical behaviour is concerned. The reason for this disagreement is not known.

To conclude, we have calculated the critical exponents associated with correlations along the $(d - m)$ directions perpendicular to the competition axes. This was possible because we introduced a constraint between the quartic loop momenta appearing in different subdiagrams in higher-loop Feynman graphs. The Lifshitz universality class turns out to reduce to the Ising-like one for the value $m = 0$ at least up to the loop order considered in this paper. In principle, the technique can be readily generalized to analyse general anisotropic Lifshitz type critical behaviour with arbitrary powers of the Laplacian in the competing directions. The study of the tricritical Lifshitz points using this formalism is also worthwhile.

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